Power expansions for the self-similar solutions of the modified Savada-Kotera equation.

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Abstract

The fourth-order ordinary differential equation, defining new transcendents, is studied. The self-similar solutions of the Kaup-Kupershmidt and Savada-Kotera equations are shown to be found taking its solutions into account. Equation studied belongs to the class of fourth-order analogues of the Painlevé equations. All the power and non-power asymptotic forms and expansions near points $z = 0$, $z = \infty$ and near arbitrary point $z = z_0$ are found by means of power geometry methods. The exponential additions to the solutions of studied equation are also determined.

1   Introduction.

We consider the fourth-order ordinary equation [1–7]

$$f(z, w) \stackrel{def}{=} w_{zzzz} - 5w^2 w_{zz} + 5w_z w_{zz} - 5ww_z^2 + w^5 - zw - \alpha = 0 \quad (1.1)$$

which is one of the fourth-order analogues of the Painlevé equations [2–7]. It belongs to the class of exactly solvable equations and possesses Bäcklund transformations, Lax pair, rational and special solutions at some values of parameter α [1–7].

More than one century ago Painlevé and his collaborators analyzed a certain class of the second-order nonlinear ordinary differential equations. Their aim was to find all the second-order canonical equations with the solutions without movable critical points and to pick out those equations which have solutions defining new special functions. As a result of investigations Painlevé and his school discovered six second-order nonlinear ordinary differential equations which solutions could not be expressed in terms of known elementary or special functions. Nowadays they go under the name of the Painlevé equations and their solutions are called the Painlevé transcendents.

In the sixtieth of the twentieth century it was shown that the Painlevé transcendents arose in the models describing physical phenomena as frequently as many other special functions [7–15]. This fact caused a significant interest to the studying of their properties and set a problem to find other nonlinear equations defining new transcendents. Equation (1.1) belongs to the class of such equations.

Let us also show that the self-similar solutions of the Kaup-Kupershmidt equation [16, 17] and the Savada-Kotera equation [17, 18] can be determined by means of ordinary differential equation (1.1).

The Kaup-Kupershmidt and the Savada-Kotera equations can be written as

$$u_t + u_{xxxxx} + 10uu_{xxx} + 10\nu u_x u_{xx} + 20u^2 u_x = 0 \tag{1.2}$$

where we consider the Kaup-Kupershmidt equation, if $\nu = 1$, and the Savada-Kotera equation, if $\nu = 5$ [19, 20].

Using the self-similar variables

$$u(z) = \frac{1}{(5t)^{2/5}} g(z), \qquad z = \frac{x}{(5t)^{1/5}} \tag{1.3}$$

we obtain

$$E_1 \equiv g_{zzzzz} + 10gg_{zzz} + 10g_z g_{zz} + 20g^2 g_z - 2g - zg_z = 0 \tag{1.4}$$

and

$$E_2 \equiv g_{zzzzz} + 10gg_{zzz} + 25g_z g_{zz} + 20g^2 g_z - 2g - zg_z = 0 \tag{1.5}$$

for the Kaup-Kupershmidt and Savada-Kotera equations accordingly.

By means of the Miura transformation

$$g = -(w_z + w^2)/2 \tag{1.6}$$

we obtain that the Kaup-Kupershmidt equation in the self-similar variables can be presented as

$$E_1 = \frac{1}{2}\left(\frac{d}{dz} + 2w\right)\frac{d}{dz}f(z, w) = 0 \tag{1.7}$$

Using the Miura transformation in the form

$$g = w_z - w^2/2 \tag{1.8}$$

we can also express the Savada-Kotera equation via equation (1.1)

$$E_2 = \left(\frac{d}{dz} - w\right)\frac{d}{dz}f(z, w) = 0 \tag{1.9}$$

So we have that the self-similar solutions of both the Kaup-Kupershmidt and Savada-Kotera equations can be determined by means of ordinary differential equation (1.1).

Painlevé equations determine the transcendental functions, so equation (1.1) is likely to do it too. However there is no exact proof of this statement. So it is important to explore the asymptotic forms and expansions of this equation.

The aim of this paper is to calculate all asymptotic forms and expansions to solutions of equation (1.1). To achieve it we use the power geometry methods [21–24].

The outline of this paper is as follows. In section 2 the general properties of equation (1.1) are discussed. In sections 3–8 asymptotic forms and expansions to the studied equation near points $z = 0$ and $z = \infty$ are given. It sections 9–12 exponential additions for these expansions are found. Finally, section 13 is devoted to asymptotic forms and power expansions to solutions of equation (1.1) near arbitrary point $z = z_0$.

2 General properties of equation (1.1).

The monomials of equation (1.1) are corresponded to points $M_1 = (-4, 1)$, $M_2 = (-2, 3)$, $M_3 = (-3, 2)$, $M_4 = (-2, 3)$, $M_5 = (0, 5)$, $M_6 = (1, 1)$ and $M_7 = (0, 0)$ (if $\alpha \neq 0$). Therefore, the carrier $S(f)$ of equation (1.1) contains points $Q_1 = M_1$, $Q_2 = M_5$, $Q_3 = M_6$, $Q_4 = M_8$ (if $\alpha \neq 0$), $Q_5 = M_3$ and $Q_6 = M_2 = M_4$.

In the case $\alpha \neq 0$ the convex hull of the carrier of equation (1.1) is the quadrangle with four apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) and four edges $\Gamma_1^{(1)} = [Q_1, Q_2]$, $\Gamma_2^{(1)} = [Q_2, Q_3]$, $\Gamma_3^{(1)} = [Q_3, Q_4]$, $\Gamma_4^{(1)} = [Q_1, Q_4]$. It is presented at fig. 1. The external normal vectors N_j ($j = 1, 2, 3, 4$) to edges $\Gamma_j^{(1)}$ ($j = 1, 2, 3, 4$) are $N_1 = (-1, 1)$, $N_2 = (4, 1)$, $N_3 = (1, -1)$, $N_4 = (-1, -4)$. They form the normal cones $U_j^{(1)}$ of edges $\Gamma_j^{(1)}$.

$$U_j^{(1)} = \{P = \lambda N_j, \ \lambda > 0\}, \ \ j = 1, 2, 3, 4. \tag{2.1}$$

The normal cones $U_j^{(0)}$ of apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3, 4$) are the angles between the edges that adjoin to the apex. They are given on fig. 2.

If $\alpha = 0$, the convex hull of the carrier of equation (1.1) is the triangle with apexes $\Gamma_j^{(0)} = Q_j$ ($j = 1, 2, 3$) and edges $\Gamma_1^{(1)} = [Q_1, Q_2]$, $\Gamma_2^{(1)} = [Q_2, Q_3]$, $\Gamma_5^{(1)} = [Q_3, Q_1]$ (fig. 3). The normal cones of equation (1.1) in this case are presented at fig. (4).
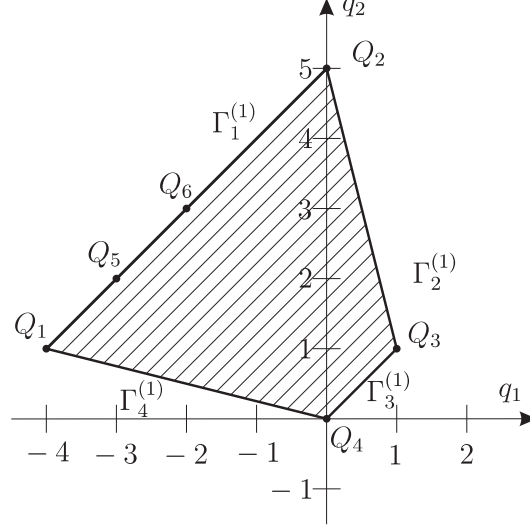
Figure 1 The carrier of equation (1.1) at $\alpha \neq 0$

Lattice $\mathbb{Z}$, where the carrier of equation (1.1) lies, is generated by basis vectors $(-4, 1)$ and $(1, 1)$.

Examining the reduced equations corresponding to bounds $\Gamma_j^{(d)}$ ($d = 0, 1$; $j = 1, 2, 3, 4$) we will obtain the expansions of equation (1.1).

The reduced equations corresponding to apexes $\Gamma_2^{(0)}$, $\Gamma_3^{(0)}$ and $\Gamma_4^{(0)}$

$$\hat{f}_2^{(0)} \overset{def}{=} w^5 = 0$$

$$\hat{f}_3^{(0)} \overset{def}{=} -zw = 0 \tag{2.2}$$

$$\hat{f}_4^{(0)} \overset{def}{=} -\alpha = 0$$

are the algebraic ones and hence they do not have non-trivial power or non-power solutions.

3 Solutions, corresponding to apex $\Gamma_1^{(0)}$.

Apex $\Gamma_1^{(0)}$ defines the following reduced equation

$$\hat{f}_1^{(0)} \overset{def}{=} w_{zzzz} = 0 \tag{3.1}$$

Let us find the reduced solutions
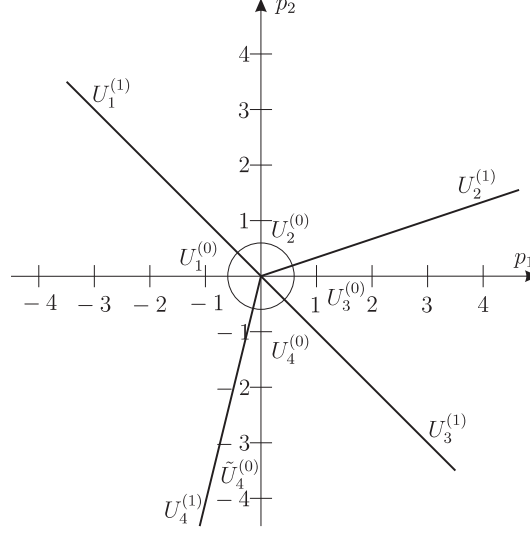
$$w = c_r z^r, \quad c_r \neq 0 \tag{3.2}$$

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Figure 2 The normal cones of equation (1.1) at $\alpha \neq 0$

for $\omega(1, r) \in U_1^{(0)}$. Since $p_1 < 0$ in the cone $U_1^{(0)}$ then $\omega = -1$, $z \to 0$ and the expansions will be the ascending power series of z. Substituting (3.2) into $\hat{f}_1^{(0)}$ and cancelling the result by z^{r-4} we get the characteristic equation

$$\chi(r) \stackrel{def}{=} r(r-1)(r-2)(r-3) = 0 \tag{3.3}$$

with four roots $r_1 = 0$, $r_2 = 1$, $r_3 = 2$, and $r_4 = 3$. Further we shall examine them separately.

Vector $R = (1, 0)$ that corresponds to the root $r_1 = 0$ being multiplied by $\omega = -1$ belongs to the cone $U_1^{(0)}$. Thus we obtain the family $\mathcal{F}_1^{(0)} 1$ of power asymptotic forms $w = c_0$ with the arbitrary constant $c_0 \neq 0$. The first variation of equation (3.1)

$$\frac{\delta \hat{f}_1^{(0)}}{\delta w} = \frac{d^4}{dz^4} \tag{3.4}$$

gives the operator

$$L(z) = \frac{d^4}{dz^4} \neq 0 \tag{3.5}$$

with the characteristic polynomial

$$\nu(k) = z^{4-k} L(z) z^k = k(k-1)(k-2)(k-3) \tag{3.6}$$

The equation $\nu(k) = 0$ has four roots $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$. As $\omega = -1$ and $r = 0$, then the cone of the problem is $\mathcal{K} = \{k > 0\}$. It contains $k_2 = 1$, $k_3 = 2$, $k_3 = 3$, which are the critical numbers.
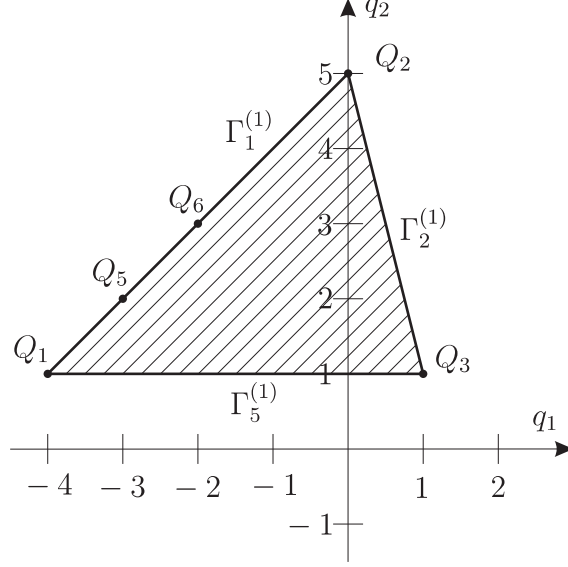
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Figure 3 The carrier of equation (1.1) at $\alpha = 0$

Thus the expansion of the solution corresponding to the reduced equation (3.1) at $r = 0$ takes the form

$$w(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \qquad (3.7)$$

where all the coefficients are constants, $c_0 \neq 0$, c_1, c_2, c_3 are the arbitrary ones and c_k ($k \geq 4$) are uniquely defined. Denote this family as $G_1^{(0)}1$. Taking into account six terms, expansion (3.7) can be presented in the form

$$w(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 +$$

$$+ \left(\alpha - 10\, c_1 c_2 - c_0{}^5 + 10\, c_2 c_0{}^2 + 5\, c_0 c_1{}^2 \right) \frac{z^4}{24} +$$

$$+ \left(c_0 - 20\, c_2{}^2 + 40\, c_2 c_0 c_1 + 30\, c_3 c_0{}^2 - 30\, c_1 c_3 + 5\, c_1{}^3 - 5\, c_0{}^4 c_1 \right) \frac{z^5}{120} + \ldots$$

In the case $r_2 = 1$ the cone of the problem is $\mathcal{K} = \{k > 1\}$. Consequently there are two critical numbers $k_2 = 2$, $k_3 = 3$. Likewise the previous case we find the family of expansions $G_1^{(0)}2$

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \qquad (3.8)$$

that is generated by the power asymptotic form $\mathcal{F}_1^{(0)}2 : \quad w = c_1 z$. Here $c_1 \neq 0$, c_2 and c_3 are the arbitrary constants.
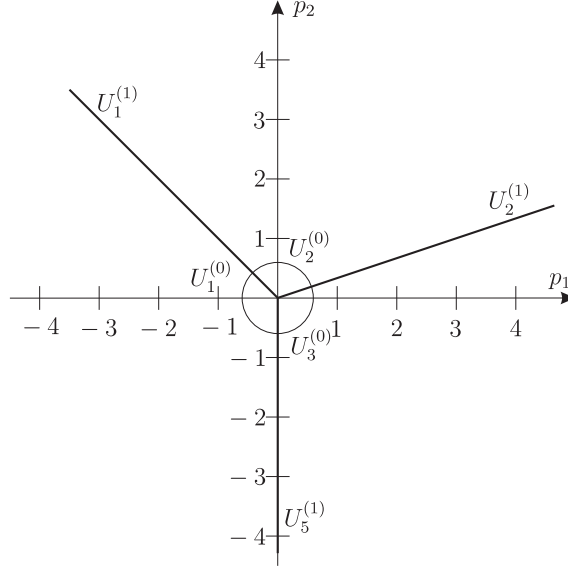
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Figure 4 The normal cones of equation (1.1) at $\alpha = 0$

For the root $r_2 = 2$ the cone of the problem is $\mathcal{K} = \{k > 2\}$. It contains $k_3 = 3$ that is the unique critical number. The power expansion corresponding to the asymptotic form $\mathcal{F}_1^{(0)}3 : \ w = c_2 \, z^2$ takes the form

$$w(z) = c_2 z^2 + c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \qquad (3.9)$$

Again $c_2 \neq 0$, c_3 are the arbitrary constants. Denote this family as $G_1^{(0)}3$.

For the root $r_3 = 3$ the cone of the problem is $\mathcal{K} = \{k > 3\}$. There are no critical numbers in this case. The expansion of the solution $G_1^{(0)}4$ corresponding to power asymptotic form $\mathcal{F}_1^{(1)}4 : \ w = c_3 \, z^3$ can be written as

$$w(z) = c_3 z^3 + \sum_{k=4}^{\infty} c_k z^k \qquad (3.10)$$

Note, that expansions (3.8), (3.9), (3.10) are the special cases of expansion (3.7).

Obtained expansions converge for sufficiently small $|z|$. The existence and the analyticity of expansions (3.7), (3.8), (3.9), (3.10) follow from the Cauchy theorem. This apex does not define non-power asymptotic forms and exponential additions.

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4 Solutions, corresponding to edge $\Gamma_1^{(1)}$.

Edge $\Gamma_1^{(1)}$ is characterized by the reduced equation

$$\hat{f}_1^{(1)}(z, w) \overset{def}{=} w_{zzzz} - 5\,w^2\,w_{zz} + 5w_z w_{zz} - 5\,w\,w_z^2 + w^5 = 0 \qquad (4.1)$$

and the normal cone $U_1^{(1)} = \{-\lambda(1, -1),\ \lambda > 0\}$. Therefore $\omega = -1$, i.e. $z \to 0$ and $r = -1$. Consequently the solution of equation (4.1) should be looked for in the form

$$w = c_{-1} z^{-1} \qquad (4.2)$$

From the determining equation

$$c_{-1}(c_{-1}^4 - 15\,c_{-1}^2 - 10c_{-1} + 24) = 0 \qquad (4.3)$$

we find the values of coefficient c_{-1} (not equal to zero): $c_{-1}^{(1)} = 1$, $c_{-1}^{(2)} = -2$, $c_{-1}^{(3)} = -3$, $c_{-1}^{(4)} = 4$. Hence we have four families of power asymptotic forms

$$\mathcal{F}_1^{(1)}1: \qquad w = z^{-1} \qquad (4.4)$$

$$\mathcal{F}_1^{(1)}2: \qquad w = -2z^{-1} \qquad (4.5)$$

$$\mathcal{F}_1^{(1)}3: \qquad w = -3z^{-1} \qquad (4.6)$$

$$\mathcal{F}_1^{(1)}4: \qquad w = 4z^{-1} \qquad (4.7)$$

Denote the families of expansions, corresponding to these asymptotic forms as $G_1^{(1)}i, \quad i = 1, 2, 3, 4$.

Let us compute the corresponding critical numbers. The first variation

$$\frac{\delta f_2^{(1)}}{\delta w} = \frac{d^4}{dz^4} - 10w_{zz}w - 5w^2\frac{d^2}{dz^2} + 5w_z\frac{d^2}{dz^2} + 5w_z z\frac{d}{dz} - 5w_z^2 - 5\,ww_z\frac{d}{dz} + 5\,w^4 \qquad (4.8)$$

applied to solutions (4.2) yields to operator

$$\mathcal{L}(z) = \frac{d^4}{dz^4} - \frac{5c_{-1}(1 + c_{-1})}{z^2}\frac{d^2}{dz^2} + \frac{10c_{-1}(1 + c_{-1})}{z^3}\frac{d}{dz} - \frac{5c_{-1}^2(5 - c_{-1}^2)}{z^4} \qquad (4.9)$$

Its characteristic polynomial is

$$\nu(k) = k^4 - 6\,k^3 + \left(11 - 5\,c_{-1} - 5\,c_{-1}^2\right)k^2 - \\ - \left(6 - 15\,c_{-1} - 15\,c_{-1}^2\right)k - 5\,c_{-1}^2\left(5 - c_{-1}^2\right) \qquad (4.10)$$

Equation $\nu(k) = 0$ has the roots:

1) $k_1 = -2$, $k_2 = 1$, $k_3 = 2$, $k_4 = 5$ for $c_{-1} = 1$;
2) $k_1 = -2$, $k_2 = 1$, $k_3 = 2$, $k_4 = 5$ for $c_{-1} = -2$;
3) $k_1 = -2$, $k_2 = -3$, $k_3 = 5$, $k_4 = 6$ for $c_{-1} = -3$;
4) $k_1 = -2$, $k_2 = -8$, $k_3 = 5$, $k_4 = 11$ for $c_{-1} = 4$.
The cone of the problem here is

$$\mathcal{K} = \{k > -1\}. \tag{4.11}$$

Thus for power asymptotic forms (4.4) and (4.5) there are three critical numbers (three roots of the characteristic polynomial belong to the cone of the problem) and for the power asymptotic forms (4.6) and (4.7) there are only two critical numbers.

The lattice, generated by basic vectors (-4,1), (1,1) and vector (-1,-1), corresponding to the shifted carrier of power asymptotic forms (4.4 – 4.7), consists of points $Q = m(-4,1) + l(1,1)$. These points intersect with the line $q_2 = -1$ if $m + l = -1$, i.e. $l = -m - 1$. As the cone of the problem here is (4.11) then

$$\mathbf{K} = \{k = -1 + 5m, \ m \in \mathbb{N}\} \tag{4.12}$$

Sets $\mathbf{K}(1)$, $\mathbf{K}(1,2)$ and $\mathbf{K}(1,2,5)$ can be written as

$$\mathbf{K}(1) = \{k = -1 + 5m + 2l, \ l, m \in \mathbb{N} \cup \{0\}, \ l + m \neq 0\} =$$
$$= \{1, 3, 4, 5, 6, 7, 8, ...\}$$

$$\mathbf{K}(1,2) = \{k = -1 + 5m + 2l + 3n, \ l, m, n \in \mathbb{N} \cup \{0\}, \ l + m + n \neq 0\} =$$
$$= \{1, 2, 3, 4, 5, 6, 7, 8, ...\}$$

$$\mathbf{K}(1,2,5) = \{k = -1 + 5m + 2l + 3n + 6j, \ j, l, m, n \in \mathbb{N} \cup \{0\}, \ j + l + m + n \neq 0\} =$$
$$= \{k, \ k \in \mathbb{N}\}$$

In this case expansion generated by family (4.4) takes the form

$$w(z) = \frac{1}{z} + \sum_{k \in \mathbb{N}} c_k^{(1)} z^k \tag{4.13}$$

The critical number 1 does not belong to the set $\mathbf{K}$ that is why the compatibility condition for c_1 holds automatically and c_1 is the arbitrary constant. The critical number 2 also does not belong to the sets $\mathbf{K}$, $\mathbf{K}(1)$ and thus c_2 is the arbitrary constant. However the critical number 5 lies in the sets $\mathbf{K}(1)$ and $\mathbf{K}(1,2)$. As a result it is necessary to verify that the compatibility condition for c_5 is true. It holds, so c_5 is the arbitrary constant too.

The three-parametric power expansion that corresponds to power asymptotic form (4.4) is as follows

$$w(z) = \frac{1}{z} + c_1 z + c_2 z^2 + \frac{1}{4} {c_1}^2 z^3 - \frac{1}{36}(1+\alpha) z^4 + c_5 z^5 +$$

$$+ \frac{1}{2880} c_1 (90\, c_2 c_1 - 7 - 25\,\alpha) z^6 +$$

$$+ \left(\frac{1}{6} c_1 c_5 + \frac{1}{18} {c_2}^2 c_1 + \frac{1}{192} {c_1}^4 + \frac{1}{540} c_2 \right) z^7 + \ldots$$

where $c_i \equiv c_i^{(1)}$, $i = 1, 2, 5$ are the arbitrary constants.

In the same way we obtain that asymptotic form (4.5) also corresponds to three-parameter power expansion

$$w(z) = -\frac{2}{z} + \sum_{k \in \mathbb{N}} c_k^{(2)} z^k \qquad (4.14)$$

Taking into account seven terms it can be presented as

$$w(z) = -\frac{2}{z} + c_1 z + c_2 z^2 - \frac{7}{2} {c_1}^2 z^3 + \left(\frac{1}{18} - \frac{1}{36}\alpha - \frac{5}{2} c_1 c_2 \right) z^4 + c_5 z^5 +$$

$$+ \frac{1}{1440} c_1 (7560\, c_1 c_2 - 71 + 40\,\alpha) z^6 +$$

$$+ \left(-\frac{2}{3} c_1 c_5 - \frac{89}{24} {c_1}^4 - \frac{19}{540} c_2 + \frac{14}{9} {c_2}^2 c_1 + \frac{1}{54} c_2 \alpha \right) z^7 + \ldots$$

where $c_i \equiv c_i^{(2)}$, $i = 1, 2, 5$ are the arbitrary constants.

Sets $\mathbf{K}(5)$, $\mathbf{K}(5,6)$ and $\mathbf{K}(5,11)$ are as follows

$$\mathbf{K}(5) = \{k = -1 + 5m + 6l,\ l, m \in \mathbb{N} \cup \{0\},\ l + m \neq 0\} =$$
$$= \{4, 5, 9, 10, 11, 14, 15, 16, 17, 19, \ldots\} \qquad (4.15)$$

$$\mathbf{K}(5,6) = \{k = -1 + 5m + 6l + 7n,\ m, l, n \in \mathbb{N} \cup \{0\},\ l + m + n \neq 0\} =$$
$$= \{4, 5, 6, 9, 10, 11, 12, 13, 14, \ldots\} \qquad (4.16)$$

$$\mathbf{K}(5,11) = \{k = -1 + 5m + 6l + 12k,\ l, m, n \in \mathbb{N} \cup \{0\},\ l + m + n \neq 0\} =$$
$$= \{4, 5, 9, 10, 11, 14, 15, 16, 17, 19, \ldots\} \qquad (4.17)$$

Using the discussion as described above, we obtain two-parametric family $G_1^{(1)}3$

$$w(z) = -\frac{3}{z} + \sum_{k \in \mathbf{K}(5,6)} c_k^{(3)} z^k \tag{4.18}$$

that can be presented as

$$w(z) = -\frac{3}{z} - \left(\frac{1}{28} - \frac{1}{84}\alpha\right) z^4 + c_5 z^5 + c_6 z^6 -$$

$$-\frac{1}{931392}(\alpha - 3)(40\alpha - 127) z^9 + \frac{1}{131040} c_5 (2307 - 755\alpha) z^{10} -$$

$$-\left(\frac{31}{156} c_5{}^2 - \frac{547}{38220} c_6 + \frac{3}{637} c_6\alpha\right) z^{11} + \dots$$

where $c_i \equiv c_i^{(3)}$, $i = 5, 6$ are the arbitrary constants, and two-parametric family $G_1^{(1)}4$

$$w(z) = \frac{4}{z} + \sum_{k \in \mathbf{K}(5,11)} c_k^{(4)} z^k \tag{4.19}$$

that can be written as

$$w(z) = \frac{4}{z} + \left(\frac{1}{126} + \frac{1}{504}\alpha\right) z^4 + c_5 z^5 +$$

$$+\frac{1}{47500992}(\alpha + 4)(25\alpha + 37) z^9 + \frac{1}{15120} c_5 (6 + 5\alpha) z^{10} +$$

$$+c_{11}z^{11} + \frac{1}{2106763997184}(\alpha + 4)\left(220\alpha^2 + 1235\alpha + 1126\right) z^{14} + \dots$$

where $c_i \equiv c_i^{(4)}$, $i = 5, 11$ are the arbitrary constants too.

Obtained expansions converge for sufficiently small $|z|$ and have no exponential additions. The reduced equation (4.1) also does not have non-power solutions.

5   Solutions, corresponding to edge $\Gamma_2^{(1)}$.

Edge $\Gamma_2^{(1)}$ is characterized by the reduced equation

$$\hat{f}_2^{(1)}(z, w) \overset{def}{=} w^5 - zw = 0 \tag{5.1}$$

and normal cone $U_2^{(1)} = \{\lambda(4, 1) = 4\lambda(1, 1/4), \lambda > 0\}$. It means that $r = 1/4$, $\omega = 1$, $z \to \infty$ and solution of this reduced equation is $w = c_{1/4}z^{1/4}$.

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Substitution this expression into equation (5.1) and cancellation the result by $z^{5/4}$ yields to determining equation $c_{1/4}(c_{1/4}^4 - 1) = 0$. Thus we have four families of power asymptotic forms

$$\mathcal{F}_2^{(1)}1: \ w = c_{1/4}^{(1)} z^{1/4}, \quad c_{1/4}^{(1)} = 1 \tag{5.2}$$

$$\mathcal{F}_2^{(1)}2: \ w = c_{1/4}^{(2)} z^{1/4}, \quad c_{1/4}^{(2)} = -1 \tag{5.3}$$

$$\mathcal{F}_2^{(1)}3: \ w = c_{1/4}^{(3)} z^{1/4}, \quad c_{1/4}^{(3)} = i \tag{5.4}$$

$$\mathcal{F}_2^{(1)}4: \ w = c_{1/4}^{(4)} z^{1/4}, \quad c_{1/4}^{(4)} = -i \tag{5.5}$$

Since the reduced equation (5.1) is algebraic one and the roots of the determining equation are simple then asymptotic forms do not have proper (and consequently critical) numbers and $\nu(k) \equiv const \neq 0$.

The shifted carrier of the power asymptotic forms (5.2) – (5.5) gives a vector $(1/4, -1)$. Points belonging to the lattice generated by this vector and the basis vectors (-4,1), (1,1) are the following $Q = (q_1, q_2) = m(1, 1) + l(1/4, -1) = (m + l/4, m - l)$ where m, l are whole numbers. At the line $q_2 = -1$ we have $q_1 = -1 + 5/4l$. Taking into consideration that the cone of the problem is $\mathcal{K} = \{k < 1/4\}$ we find the set $\mathbf{K}$

$$\mathbf{K} = \{k = -1 - 5l/4, \ l \in \mathbb{N} \cup \{0\}\} \tag{5.6}$$

The expansions to solutions can be written as

$$G_2^{(1)}n: \ w(z) = \varphi^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \sum_{l=0}^{\infty} c_{-1-5l/4}^{(n)} z^{-1-5l/4} \tag{5.7}$$

In this expression coefficients $c_{-1-5l/4}^{(n)}$ can be sequentially computed. The calculation of the coefficients c_{-1} yields to $c_{-1} = \alpha/4$. Taking into account five terms, we obtain

$$w(z) = \varphi^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \frac{1}{4}\frac{\alpha}{z} - \frac{5}{32}\frac{(c_{1/4}^{(n)})^3 (1 + \alpha^2)}{z^{9/4}} +$$

$$+\frac{5}{256}\frac{(c_{1/4}^{(n)})^2 (8\alpha^3 + 29\alpha + 3)}{z^{7/2}} - \frac{1}{2048}\frac{c_{1/4}^{(n)} (365\alpha^4 + 3210\alpha^2 + 590\alpha + 2013)}{z^{19/4}} + \ldots$$

The obtained expansions seem to be divergent ones.

Non-power asymptotic forms do not correspond to edge $\Gamma_2^{(1)}$ but it generates exponential additions which will be computed later.

6 Solutions, corresponding to edge $\Gamma_3^{(1)}$.

Edge $\Gamma_3^{(1)}$ exists if $\alpha \neq 0$. It is characterized by the reduced equation

$$\hat{f}_3^{(1)}(z, w) \stackrel{def}{=} -zw - \alpha = 0 \tag{6.1}$$

and the normal cone $U_3^{(1)} = \{\lambda(1, -1), \lambda > 0\}$. In this case $r = -1$, $\omega = 1$, $z \longrightarrow \infty$ and power asymptotic can be presented in the form

$$\mathcal{F}_3^{(1)} 1 : \qquad w = \frac{c_{-1}}{z}, \qquad c_{-1} = -\alpha \tag{6.2}$$

As equation (6.1) is algebraic one and

$$\nu(k) = z^{-1} \frac{\delta \hat{f}_3^{(1)}}{\delta y} = -1 \tag{6.3}$$

then the solutions of equation (6.1) do not have critical numbers. The cone of the problem is $\mathcal{K} = \{k < -1\}$. The shifted carrier of the power asymptotic (6.2) gives the vector $(-1, -1)$, which belongs to the lattice generated by the carrier of the studied equation. So we obtain the set $\mathbf{K}$

$$\mathbf{K} = \{k = -1 - 5m, \ m \in \mathbb{N}\} \tag{6.4}$$

Therefore we can determine the power expansion, corresponding to the asymptotic form (6.2)

$$G_3^{(1)} : \quad w(z) = \widetilde{\varphi}(z) = \frac{c_{-1}}{z} + \sum_{m=1}^{\infty} c_{-1-5m} \, z^{-1-5m} \tag{6.5}$$

In this expression all coefficients can be sequentially found. Taking into account three terms, it can be rewritten as

$$w(z) = \widetilde{\varphi}(z) = -\frac{\alpha}{z} - \frac{P}{z^6} - \frac{P \left(5\,\alpha^4 - 295\,\alpha^2 + 270\alpha + 3024\right)}{z^{11}} + \dots$$

where

$$P = \alpha(\alpha + 1)(\alpha - 2)(\alpha - 3)(\alpha + 4) \tag{6.6}$$

and all other not written out coefficients are proportional to this factor. Hence if $P = 0$ this expansion determines the simplest exact rational solutions of equation (1.1).

Edge $\Gamma_3^{(1)}$ does not generate non-power asymptotic forms but it defines exponential additions, which will be found below.

7 Solutions, corresponding to edge $\Gamma_4^{(1)}$.

Edge $\Gamma_4^{(1)}$ exists, if $\alpha \neq 0$. It defines the following reduced equation

$$\hat{f}_4^{(1)}(z, w) \overset{def}{=} w_{zzzz} - \alpha = 0 \tag{7.1}$$

and the normal cone $U_4^{(1)} = \{-\lambda(1, 4), \ \lambda > 0\}$. Thus $\omega = -1$, i.e. $z \to 0$, $r = 4$ and we have the unique family of power asymptotic forms

$$\mathcal{F}_4^{(1)}1: \quad w = c_4 z^4, \qquad c_4 = \frac{\alpha}{24} \tag{7.2}$$

Let us find the critical numbers. The first variation of equation (7.1) is

$$\frac{\delta \hat{f}_4^{(1)}}{\delta w} = \frac{d^4}{dz^4} \tag{7.3}$$

The proper numbers are $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$. None of them belongs to the cone of the problem $\mathcal{K} = \{k > 4\}$. Consequently there are no critical numbers here.

Vector, corresponding to the shifted carrier of asymptotic form (7.2) is (4,-1), so it belongs to the lattice generated by the basis vectors. Therefore set $\mathbf{K}$ is

$$\mathbf{K} = \{k = 4 + 5m, \quad m \in \mathbb{N}\} \tag{7.4}$$

and the power expansion can be written as

$$w(z) = z^4 \left(\frac{\alpha}{24} + \sum_{m=1}^{\infty} c_{4+5m} \, z^{5m} \right) \tag{7.5}$$

All the coefficients can be uniquely computed. Taking into account three terms, we can write this expansion $G_4^{(1)}1$ as

$$w(z) = \frac{1}{24} \alpha \, z^4 - \frac{(10\,\alpha - 1)}{72576} \alpha \, z^9 + \frac{(3120\,\alpha^2 - 185\,\alpha + 2)}{3487131648} \alpha \, z^{14} + \dots$$

The obtained expansion can be considered as the special case of expansion (3.7) at $c_0 = c_1 = c_2 = c_3 = 0$. It converges for sufficiently small $|z|$.

Edge $\Gamma_4^{(1)}$ does not define exponential additions and non-power asymptotic forms.

8 Solutions, corresponding to edge $\Gamma_5^{(1)}$.

Edge $\Gamma_5^{(1)}$ exists, if and only if $\alpha = 0$.

The reduced equation which corresponds to the edge $\Gamma_5^{(1)}$, takes the form

$$\hat{f}_2^{(5)}(z, w) \stackrel{def}{=} w_{zzzz} - zw = 0 \tag{8.1}$$

It does not possess solutions in the form $w = c_r z^r$, except of the trivial one $w \equiv 0$. But edge $\Gamma_5^{(1)}$ defines non-power asymptotic forms of equation (1.1).

As edge $\Gamma_5^{(1)}$ is the horizontal one, we can use the logarithmic transformation

$$y = \frac{d \ln w}{dz} \tag{8.2}$$

Hence we have the relations

$$w' = yw, \quad w'' = (y' + y^2)w, \quad w''' = (y'' + 3yy' + y^3)w,$$
$$w'''' = (y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4)w$$

After application this transformation and after cancellation of the result by w equation (8.1) can be rewritten as

$$h(z, y) \stackrel{def}{=} y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 - z = 0 \tag{8.3}$$

The carrier $S(h)$ of this equation consists of the following points: $M_1 = (-3, 1)$, $M_2 = (0, 4)$, $M_3 = (1, 0)$, $M_5 = (-2, 2)$ and $M_6 = (-1, 3)$.

Their convex hull $\Gamma(h)$ is the triangle (fig. 5). The normal cones of its bounds are presented at fig. 6.

According to [23], the cone of the problem here is $p_1 + p_2 \geq 0$, $p_1 \geq 0$. It intersects with the normal cones $\tilde{U}_2^{(0)}$, $\tilde{U}_3^{(0)}$, $\tilde{U}_2^{(1)}$ only. So we should examine the reduced equations which correspond to the bounds $\tilde{\Gamma}_2^{(0)}$, $\tilde{\Gamma}_3^{(0)}$, $\tilde{\Gamma}_2^{(1)}$.

Apexes $\tilde{\Gamma}_2^{(0)}$ and $\tilde{\Gamma}_3^{(0)}$ are characterized by the algebraic reduced equations $y^4 = 0$ and $-z = 0$ accordingly, so they do not give suitable solutions.

Edge $\tilde{\Gamma}_2^{(1)}$ defines algebraic reduced equation

$$y^4 - z = 0 \tag{8.4}$$

which has four suitable solutions $y^{(n)}(z) = c_{1/4}^{(n)} z^{1/4}$, where $c_{1/4}^{(1,2)} = \pm 1$, $c_{1/4}^{(3,4)} = \pm i$, $r = 1/4$ and $\omega = 1 \Rightarrow z \to \infty$. These solutions do not have critical numbers.

Using the inversion to transformation (8.2), we obtain four weak asymptotic forms for solutions of equation (1.1) near $z \to \infty$

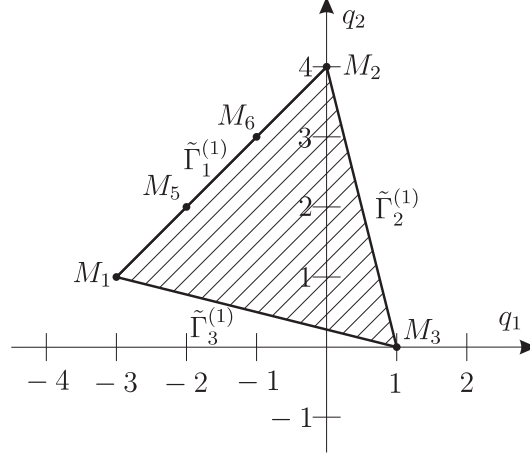$$w(z) = C_1 \exp\left(\frac{4}{5}c_{1/4}^{(n)} z^{5/4}\right)$$

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Figure 5 The carrier of equation (8.3).

where $n = 1, 2, 3, 4$ and C_1 is the arbitrary constant.

Now let us obtain the strong asymptotic forms.

The lattice of the shifted carrier of equation (8.3) is generated by vectors $(1/4, -1)$ and $(1, 1)$. Taking into account the cone of the problem $\mathcal{K} = \{k < 1/4\}$, we find set $\mathbf{K}$

$$\mathbf{K} = \{k = 1/4 - 5l/4, \quad l \in \mathbb{N}\} \tag{8.5}$$

So equation (8.3) has solutions in the form

$$y^{(n)}(z) = c_{1/4}^{(n)} z^{1/4} + \sum_{l=1}^{\infty} c_{1/4-5l/4}^{(n)} z^{1/4-5l/4}, \quad n = 1, 2, 3, 4 \tag{8.6}$$

where the coefficients $c_{1/4-5l/4}^{(n)}$ are uniquely defined. The coefficient $c_{-1}^{(n)}$ does not depend on n: $c_{-1}^{(n)} = -3/8$. Taking into account transformation (8.2), we get

$$\ln w = \int y\,dz = \frac{4}{5} c_{1/4}^{(n)} z^{5/4} + C_0 - \frac{3}{8} \ln z + \sum_{l=2}^{\infty} \frac{4 c_{1/4-5l/4}^{(n)}}{5(1-l)} z^{5(1-l)/4} \tag{8.7}$$

In this expression C_0 is a constant of integration. So we obtain

$$w(z) = \frac{C_1}{z^{3/8}} \exp\left[\frac{4}{5} c_{1/4}^{(n)} z^{5/4} + \sum_{l=2}^{\infty} \frac{4 c_{1/4-5l/4}^{(n)}}{5(1-l)} z^{5(1-l)/4}\right] \tag{8.8}$$
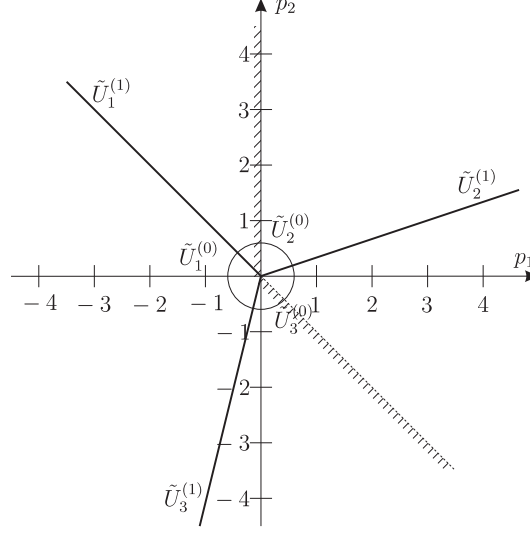
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Figure 6 The normal cones of equation (8.3).

Taking into account three terms of the series, the obtained non-power strong asymptotic forms of equation (1.1), existing under $\alpha = 0$, can be written as

$$w(z) = \frac{C_1}{z^{3/8}} \exp \left[\frac{4}{5} c_{1/4}^{(n)} z^{5/4} + \frac{9}{32} \frac{1}{c_{1/4}^{(n)} z^{5/4}} + \frac{45}{256} \frac{1}{(c_{1/4}^{(n)})^2 z^{5/2}} + \dots \right]$$

9 Exponential additions of the first level for expansions, corresponding to edge $\Gamma_2^{(1)}$.

Let us find exponential additions of the first level for expansions (5.7), corresponding to edge $\Gamma_2^{(1)}$.

We are looking for solutions in the form

$$w(z) = \varphi(z) + u(z) \tag{9.1}$$

Here and later we use the notation $\varphi = \varphi^{(n)}$, $u = u^{(n)}$, $n = 1, 2, 3, 4$.

The reduced equation for the addition $u(z)$ is a linear equation

$$M^{(1)}(z)u(z) = 0 \tag{9.2}$$

where $M^{(1)}(z)$ is the first variation of equation (1.1) at solution $w(z) = \varphi(z)$. As long as

$$\frac{\delta f}{\delta w} = \frac{d^4}{dz^4} - 5w^2 \frac{d^2}{dz^2} - 10ww_{zz} + 5w_z \frac{d^2}{dz^2} + 5w_{zz} \frac{d}{dz} - 10ww_z \frac{d}{dz} - 5w_z^2 + 5w^4 - z \tag{9.3}$$

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then

$$M^{(1)}(z) = \frac{d^4}{dz^4} - 5(\varphi^2 - \varphi_z)\frac{d^2}{dz^2} + 5(\varphi_{zz} - 2\varphi\varphi_z)\frac{d}{dz} - 5\varphi_z^2 - 10\varphi\varphi_{zz} + 5\varphi^4 - z \tag{9.4}$$

and equation (9.2) can be rewritten as

$$\frac{d^4u}{dz^4} - 5(\varphi^2 - \varphi_z)\frac{d^2u}{dz^2} + 5(\varphi_{zz} - 2\varphi\varphi_z)\frac{du}{dz} - \\ - [5\varphi_z^2 + 10\varphi\varphi_{zz} - 5\varphi^4 + z]u = 0 \tag{9.5}$$

Assumed that

$$\zeta(z) = \frac{d\ln u(z)}{dz} \tag{9.6}$$

we have

$$\frac{du}{dz} = \zeta u, \qquad \frac{d^2u}{dz^2} = \zeta_z u + \zeta^2 u, \qquad \frac{d^3u}{dz^3} = \zeta_{zz}u + 3\zeta\zeta_z u + \zeta^3 u,$$

$$\frac{d^4u}{dz^4} = \zeta_{zzz}u + 4\zeta\zeta_{zz} + 3\zeta_z^2 u + 6\zeta^2\zeta_z u + \zeta^4 u$$

Using these expansions, from equation (9.5) we obtain

$$\zeta_{zzz} + 4\zeta\zeta_{zz} + 3\zeta_z^2 + 6\zeta^2\zeta_z + \zeta^4 - 5(\varphi^2 - \varphi_z)(\zeta_z + \zeta^2) + \\ + 5\zeta(\varphi_{zz} - 2\varphi\varphi_z) + 5\varphi^4 - 5\varphi_z^2 - 10\varphi\varphi_{zz} - z = 0 \tag{9.7}$$

The carrier of this equation consists of points

$$Q_{k,0} = \left(1 - \frac{5}{4}k, 0\right), \qquad Q_{k,1} = \left(-\frac{1}{2} - \frac{5}{4}k, 1\right),$$

$$Q_{k,2} = \left(\frac{1}{2} - \frac{5}{4}k, 2\right), \qquad Q_{0,3} = (-1, 3), \qquad Q_{0,4} = (0, 4), \tag{9.8}$$

$$k \in \mathbb{N} \cup \{0\}$$

where the doubled index in the points numeration is introduced for the notation convenience.

The convex hull of these points is the string, presented at fig. 7.

It should examine the edge, passing through points $Q_{0,0} = (1, 0)$, $Q_{0,2} = (1/2, 2)$ and $Q_{0,4} = (0, 4)$, the external normal here is $N = (4, 1)$. This edge is corresponded by reduced equation

$$h_1^{(1)}(z, \zeta) \stackrel{def}{=} \zeta^4 - 5c_{1/4}^2\sqrt{z}\zeta^2 + (5c_{1/4}^4 - 1)z = 0 \tag{9.9}$$
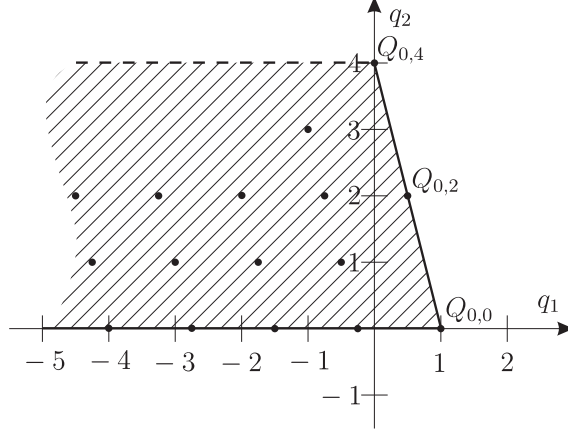
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Figure 7 The carrier of equation (9.7)

where $c_{1/4}$ is the highest coefficient of expansion $\varphi(z)$ and can have four possible values $c_{1/4}^{(n)}$, $n = 1, 2, 3, 4$ (see (5.2)-(5.5)). Taking into account, that $\forall n \Rightarrow (c_{1/4}^{(n)})^4 = 1$, for each value of $c_{1/4}$ we obtain four solutions of equation (9.9)

$$\zeta(z) = g_{1/4} z^{1/4} \tag{9.10}$$

where

$$g_{1/4} \equiv g_{1/4}^{(n;\ l=\{1,2,3,4\})} = \pm c_{1/4}^{(l)} \sqrt{\frac{5 \pm 3}{2}}$$

Reduced equation (9.9) is algebraic one, so it does not have critical numbers.

The shifted carrier of reduced solutions (9.10) gives the vector $(-1/4, 1)$, which belongs to the lattice, generated by the points of the carrier of equation (9.7). The basis vectors of this lattice are $(5/4, 0)$ and $(1, 1)$. So we can present the points of this lattice in the form

$$Q = (q_1, q_2) = k(1, 1) + m \left(\frac{5}{4}, 0 \right) = \left(k + \frac{5m}{4}, k \right)$$

At the line $q_2 = -1$ we have $k = -1$ and hence $q_1 = -1 + 5m/4$. Since the cone of the problem is $\mathcal{K} = \left\{ k < \frac{1}{4} \right\}$, set $\mathbf{K}$ can be written as

$$\mathbf{K} = \left\{ \frac{1 - 5k}{4}, k \in \mathbb{N} \right\} \tag{9.11}$$

Power expansions to solutions of equation (9.7) are

$$\zeta(z) \equiv \zeta^{(n,l)}(z) = g_{1/4}^{(n,l)} z^{1/4} + \sum_{k \in \mathbb{N}} g_{(1-5k)/4}^{(n,l)} z^{(1-5k)/4}, \qquad n, l = 1, 2, 3, 4 \tag{9.12}$$

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where coefficients $g^{(n,\,l)}_{(1-5k)/4}$, $k \in \mathbb{N}$ can be uniquely determined.

For example, coefficient $g^{1/4}_{-1}$ can be expressed as

$$g^{1/4}_{-1} = \frac{10\alpha s(s^2 - 2) - 5\,s^3 - 15\,s^2 + 24}{8(5s^2 - 8)}, \qquad s = \frac{g_{1/4}}{c_{1/4}}$$

that for different values of $g^{(n)}_{1/4}$ gives values $(5\alpha - 2)/12$, $(-5\alpha - 7)/12$, $(10\alpha - 19)/24$ or $(-10\alpha + 1)/24$.

Using the inverse transformation to (9.6)

$$u(z) = C_1 \exp \int \zeta(z)dz \tag{9.13}$$

we can compute the exponential additions

$$u(z) \equiv u^{(n,\,l)}(z) = C_1\, z^{g^{(n,\,l)}_{-1}} \exp\left[\frac{4}{5}\, g^{(n,\,l)}_{1/4}\, z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} g^{(n,\,l)}_{(1-5k)/4} z^{5(1-k)/4}\right]$$

$$n, l = 1, 2, 3, 4$$

$$\tag{9.14}$$

Here C_1 and later C_2 and C_3 are constants of integration.

So for each expansion $G^{(1)}_2 n$ we have found four one-parametric families of additions $G^{(1)}_2 n\, G^{(1)}_1 l$ $(n, l = 1, 2, 3, 4)$.

Additions $u^{(n,\,l)}(z)$ are exponentially small at $z \to \infty$ in those sectors of the complex plane z, where

$$Re\left[g^{(n,\,l)}_{1/4}\, z^{5/4}\right] < 0 \tag{9.15}$$

10 Exponential additions of the second level, corresponding to edge $\Gamma^{(1)}_2$.

In this section we compute the exponential additions of the second level $v(z)$, i.e. the additions to the solutions $\zeta(z)$. The reduced equation for this addition takes the form

$$M^{(2)}(z)v(z) = 0 \tag{10.1}$$

where operator $M^{(2)}(z)$ is the first variation of (9.7) at solutions $\zeta(z)$. Equation (10.1) can be rewritten as

$$\frac{d^3v}{dz^3} + 4\zeta\frac{d^2v}{dz^2} + (6\zeta_z + 6\zeta^2 - 5\varphi^2 + 5\varphi_z)\frac{dv}{dz}+$$

$$+ (4\zeta_{zz} + 12\zeta\zeta_z + 4\zeta^3 - 10\varphi^2\zeta + 10\varphi_z\zeta - 10\varphi\varphi_z + 5\varphi_{zz})v = 0 \tag{10.2}$$

Making the transformation of variables

$$\frac{d\ln v}{dz} = \xi,$$ (10.3)

we have

$$\frac{dv}{dz} = \xi v, \qquad \frac{d^2 v}{dz^2} = \xi_z v + \xi^2 v, \qquad \frac{d^3 v}{dz^3} = \xi_{zz} v + 3\xi\xi_z v + \xi^3 v$$

and equation (10.2) transforms to

$$\xi_{zz} + 3\xi\xi_z + \xi^3 + 4\zeta(\xi_z + \xi^2) + (6\zeta_z + 6\zeta^2 - 5\varphi^2 + 5\varphi_z)\xi +$$
$$+ 4\zeta_{zz} + 12\zeta\zeta_z + 4\zeta^3 - 10\varphi^2\zeta + 10\varphi_z\zeta - 10\varphi\varphi_z + 5\varphi_{zz} = 0 \quad (10.4)$$

The carrier of this equation consists of points

$$Q_{0,3} = (0,3), \quad Q_{k,2} = \left(\frac{1}{4} - \frac{5}{4}k,\, 2\right), \quad Q_{k,1} = \left(\frac{1}{2} - \frac{5}{4}k,\, 1\right),$$
$$Q_{k,0} = \left(\frac{3}{4} - \frac{5}{4}k,\, 1\right), \qquad k \in \mathbb{N} \cup \{0\}$$ (10.5)

Their convex hull is the string, presented at fig. 8. To obtain the exponential additions it should examine the edge, passing through points $Q_{0,3} = (0,3)$, $Q_{0,2} = (1/4, 2)$, $Q_{0,1} = (1/2, 1)$ and $Q_{0,0} = (3/4, 0)$. The reduced equation corresponding to this edge is

$$\xi^3 + 4g_{1/4}z^{1/4}\xi^2 + [6g_{1/4}^2 - 5c_{1/4}^2]z^{1/2}\xi + [4g_{1/4}^3 - 10g_{1/4}c_{1/4}^2]z^{3/4} = 0 \quad (10.6)$$
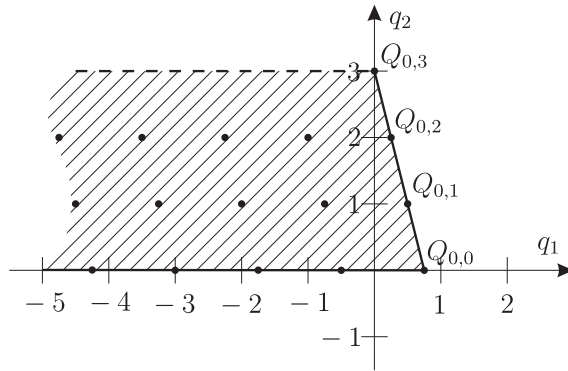

Figure 8 The carrier of equation (10.4)

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Solution of equation (10.6) can be presented in the form

$$\xi(z) = r_{1/4} z^{1/4} \tag{10.7}$$

where $r_{1/4}$ is one of the roots of the cubic equation

$$r^3 + 4\,g_{1/4} r^2 + \left(6\,g_{1/4}^2 - 5\,c_{1/4}^2\right) r + 4\,g_{1/4}^3 - 10\,g_{1/4}\,c_{1/4}^2 = 0. \tag{10.8}$$

and, therefore, also depends on $c_{1/4} = c_{1/4}^n$ and $g_{1/4}^{n,l}$, $n, l = 1, 2, 3, 4$. Solving this equation we obtain

$$r_{1/4}^{(n,l,m=1)} = -2\,g_{1/4}, \qquad r_{1/4}^{(n,l,m=\{2,3\})} = -g_{1/4} \pm \sqrt{5\,c_{1/4}^2 - g_{1/4}^2}$$

Lattice, corresponding to the shifted carrier of equation (10.4), can be generated by vectors $(1, 1)$ and $(5/4, 0)$. Vector $(-1/4, 1)$, conforming to the shifted carrier of reduced solutions (10.7), belongs to this lattice. So set **K** coincides with (9.11) and power expansions of functions $\xi^{(n,l,m)}(z)$ can be written as

$$\xi(z) \equiv \xi^{(n,l,m)}(z) = r_{1/4}^{(n,l,m)} z^{1/4} + \sum_{k \in \mathbb{N}} r_{(1-5k)/4}^{(n,l,m)} z^{(1-5k)/4},$$

$$n, l = 1, 2, 3, 4; \qquad m = 1, 2, 3 \tag{10.9}$$

where coefficients $r_{(1-5k)/4}^{(n,l,m)}$, $k \in \mathbb{N}$ can be uniquely determined.

So we have found exponential additions $v^{(n,l,m)}(z)$ to solutions $\zeta^{(n,l)}(z)$

$$v(z) \equiv v^{(n,l,m)}(z) = C_2\, z^{r_{-1}^{(n,l,m)}} \exp\left[\frac{4}{5} r_{1/4}^{(n,l,m)} z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} r_{(1-5k)/4}^{(n,l,m)} z^{5(1-k)/4}\right]$$

$$n = 1, 2, 3, 4; \qquad l = 1, 2, 3, 4; \qquad m = 1, 2, 3 \tag{10.10}$$

They are exponentially small at $z \to \infty$ on condition that

$$Re\left[r_{1/4}^{(n,l,m)} z^{5/4}\right] < 0 \tag{10.11}$$

11 Exponential additions of the third level, corresponding to edge $\Gamma_2^{(1)}$.

In this section we are looking for exponential additions of the third level $\theta(z)$, i.e. additions to the solutions $\xi(z)$. The reduced equation for addition $\theta(z)$ is the following

$$M^{(3)}(z)\theta(z) = 0 \tag{11.1}$$

Operator $M^{(3)}(z)$ can be found as the first variation of (10.4) at solutions $\xi(z)$ and then equation (11.1) for function $\theta(z)$ can be rewritten as

$$\theta_{zz} + (3\xi + 4\zeta)\theta_z + (3\xi_z + 3\xi^2 + 8\xi\zeta + 6\zeta_z + 6\zeta^2 - 5\varphi^2 + 5\varphi_z)\theta = 0 \quad (11.2)$$

Using new variable η, satisfying the relation

$$\frac{d\ln\theta}{dz} = \eta \qquad (11.3)$$

and taking into account that

$$\frac{d\theta}{dz} = \eta\theta, \qquad \frac{d^2\theta}{dz^2} = \eta_z\theta + \eta^2\theta$$

we obtain, that equation (11.2) transfers to equation

$$\eta_z + \eta^2 + (3\xi + 4\zeta)\eta + 3\xi_z + 3\xi^2 + 8\xi\zeta + 6\zeta_z + 6\zeta^2 - 5\varphi^2 + 5\varphi_z = 0 \quad (11.4)$$

The carrier of this equation is composed of points

$$Q_{0,2} = (0,2), \quad Q_{k,1} = \left(\frac{1}{4} - \frac{5}{4}k, 1\right), \quad Q_{k,0} = \left(\frac{1}{2} - \frac{5}{4}k, 1\right), \qquad k \in \mathbb{N}\cup\{0\}$$
$$(11.5)$$

The convex hull of these points is the string, presented at fig. 9. To obtain the exponential additions it should examine the edge, passing through points $Q_{0,2} = (0,2)$, $Q_{0,1} = (1/4, 1)$ and $Q_{0,0} = (1/2, 0)$. Reduced equation, corresponding to this edge, can be written as

$$\eta^2 + (3r_{1/4} + 4g_{1/4})z^{1/4}\eta + \left(3r_{1/4}^2 + 8r_{1/4}g_{1/4} + 6g_{1/4}^2 - 5c_{1/4}^2\right)z^{1/2} = 0$$
$$(11.6)$$

where $c_{1/4} \equiv c_{1/4}^{(n)}$, $g_{1/4} \equiv g_{1/4}^{(n,l)}$, $r_{1/4} \equiv r_{1/4}^{(n,l,m)}$, $n, l = 1, 2, 3, 4$, $m = 1, 2, 3$.

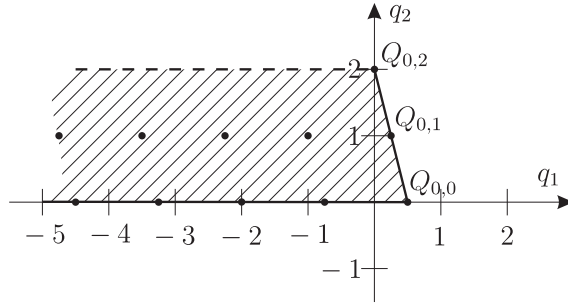Figure 9 The carrier of equation (11.4)

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Solution of equation (11.6) can be presented as

$$\eta(z) = q_{1/4} z^{1/4} \tag{11.7}$$

where $q_{1/4}$ is one of the roots of the quadratic equation

$$q^2 + (3r_{1/4} + 4g_{1/4})q + 3r_{1/4}^2 + 8r_{1/4}g_{1/4} + 6g_{1/4}^2 - 5c_{1/4}^2 = 0 \tag{11.8}$$

Solving this equation we obtain

$$q_{1/4}^{(n,l,m,p=\{1,2\})} = -\frac{3}{2} r_{1/4} - 2\, g_{1/4} \pm \frac{1}{2}\sqrt{20\, c_{1/4}^2 - 3\, r_{1/4}^2 - 8\, r_{1/4}\, g_{1/4} - 8\, g_{1/4}^2}$$

The lattice of the shifted carrier of equation (11.4) can be generated by vectors $(1, 1)$, and $(5/4, 0)$. So set $\mathbf{K}$ coincides with (9.11). Power expansions for $\eta^{(n,l,m,p)}(z)$ can be presented as

$$\eta(z) \equiv \eta^{(n,l,m,p)}(z) = q_{1/4}^{(n,l,m,p)} z^{1/4} + \sum_{k \in \mathbb{N}} q_{(1-5k)/4}^{(n,l,m,p)}\, z^{(1-5k)/4},$$

$$n, l = 1, 2, 3, 4; \ m = 1, 2, 3; \ p = 1, 2 \tag{11.9}$$

where coefficients $q_{(1-5k)/4}^{(n,l,m,p)}$, $k \in \mathbb{N}$ can be uniquely determined.

So, exponential additions $y^{(n,l,m,p)}(z)$ to solutions $\xi^{(n,l,m)}(z)$ can be written as

$$\theta(z) \equiv \theta^{(n,l,m,p)}(z) =$$

$$= C_3\, z^{q_{-1}^{(n,l,m,p)}} \exp\left[\frac{4}{5} q_{1/4}^{(n,l,m,p)}\, z^{5/4} + \sum_{k=2}^{\infty} \frac{4}{5(1-k)} q_{(1-5k)/4}^{(n,l,m,p)} z^{5(1-k)/4}\right]$$

$$n, l = 1, 2, 3, 4; \quad m = 1, 2, 3; \quad p = 1, 2 \tag{11.10}$$

They are exponentially small at $z \to \infty$ if

$$Re\left[q_{1/4}^{(n,l,m,p)}\, z^{5/4}\right] < 0 \tag{11.11}$$

Thus for expansion (5.7) to solutions of equation (1.1) near $z = \infty$ three levels of exponential additions have been found. Therefore, solutions $w(z)$ at $z \to \infty$ can be written as

$$w(z) = \varphi^{(n)}(z) +$$

$$+\exp\left[\int dz \left(\zeta^{(n,l)}(z) + \exp\left[\int dz \left(\xi^{(n,l,m)}(z) + \exp\left(\int dz\, \eta^{(n,l,m,p)}(z)\right)\right)\right]\right)\right]$$

where $n, l = 1, 2, 3, 4; \ m = 1, 2, 3; \ p = 1, 2$.

12 Three levels of exponential additions, corresponding to edge $\Gamma_3^{(1)}$.

Let us find the exponential addition of the first level $u(z)$ to expansion (6.5), i.e.

$$w(z) = \widetilde{\varphi}(z) + \widetilde{u}(z) \tag{12.1}$$

To obtain this addition we use the technique, described in section 9.

Using new variable

$$\widetilde{\zeta}(z) = \frac{d \ln \widetilde{u}(z)}{dz} \tag{12.2}$$

we obtain equation for $\widetilde{\zeta}(z)$

$$\widetilde{\zeta}_{zzz} + 4\widetilde{\zeta}\widetilde{\zeta}_{zz} + 3\widetilde{\zeta}_z^2 + 6\widetilde{\zeta}^2\widetilde{\zeta}_z + \widetilde{\zeta}^4 - 5(\widetilde{\varphi}^2 - \widetilde{\varphi}_z)(\widetilde{\zeta}_z + \widetilde{\zeta}^2) +$$
$$+5\widetilde{\zeta}(\widetilde{\varphi}_{zz} - 2\widetilde{\varphi}\widetilde{\varphi}_z) + 5\widetilde{\varphi}^4 - 5\widetilde{\varphi}_z^2 - 10\widetilde{\varphi}\widetilde{\varphi}_{zz} - z = 0 \tag{12.3}$$

The carrier of this equation consists of points

$$Q_{k,0} = (1 - 5k, 0), \quad Q_{k,1} = (-3 - 5k, 1),$$
$$Q_{k,2} = (-2 - 5k, 2), \quad Q_{0,3} = (-1, 3), \quad Q_{0,4} = (0, 4), \tag{12.4}$$
$$k \in \mathbb{N} \cup \{0\}$$

The convex hull of these points is the string, similar to one, presented at fig. 7.

The reduced equation, corresponding to the edge, passing through points $Q_{0,0} = (1,0)$ and $Q_{0,4} = (0,4)$, can be written as

$$h_1^{(1)}(z, \widetilde{\zeta}) \stackrel{def}{=} \widetilde{\zeta}^4 - z = 0 \tag{12.5}$$

This equation has four solutions in the form

$$\widetilde{\zeta}(z) \equiv \widetilde{g}_{1/4} z^{1/4} \tag{12.6}$$

where $\widetilde{g}_{1/4}^4 = 1$, i.e. $\widetilde{g}_{1/4}^{(l=\{1,2\})} = \pm 1$, $\widetilde{g}_{1/4}^{(l=\{3,4\})} = \pm i$.

The shifted carrier of equation (12.3) lies in the lattice with the basis vectors $(1, 1)$ and $(0, 5)$. Together with vector (-1/4,1), corresponding to the shifted carrier of reduced solutions (12.6), they generate the new lattice with the basis vectors $(-1/4, 1)$ and $(0, 5)$. The cone of the problem here is $\mathcal{K} = \left\{ k < \frac{1}{4} \right\}$, then we get

$$\mathbf{K} = \left\{ \frac{1 - 5k}{4}, k \in \mathbb{N} \right\} \tag{12.7}$$

Hence power expansions, conforming to reduced solutions (12.6) can be written as

$$\widetilde{\zeta}(z) \equiv \widetilde{\zeta}^{(l)}(z) = \widetilde{g}_{1/4}^{(l)} z^{1/4} + \sum_{k \in \mathbb{N}} \widetilde{g}_{(1-5k)/4}^{(l)} z^{(1-5k)/4}, \qquad l = 1, 2, 3, 4 \quad (12.8)$$

where coefficients $\widetilde{g}_{(1-5k)/4}^{(l)}$, $k \in \mathbb{N}$ can be uniquely determined.

Taking into account three terms, we can write

$$\widetilde{\zeta}^{(l)}(z) = \widetilde{g}_{1/4}^{(l)} z^{1/4} - \frac{3}{8} z^{-1} + \frac{5}{128} \frac{32\,\alpha^2 - 32\,\alpha - 9}{\widetilde{g}_{1/4}^{(l)}} z^{-9/4} + \dots$$

As the process of calculations of the exponential additions of the second and third levels coincides with that described in sections 10 and 11, let us just list the results here.

The second level of additions takes the form

$$\widetilde{\xi}(z) \equiv \widetilde{\xi}^{(l,m)}(z) = \widetilde{r}_{1/4}^{(l,m)} z^{1/4} + \frac{1}{4} z^{-1} + \sum_{k \in \mathbb{N}} \widetilde{r}_{-1-5k/4}^{(l,m)} z^{-1-5k/4},$$
$$l = 1, 2, 3, 4; \qquad m = 1, 2, 3 \tag{12.9}$$

where coefficient $\widetilde{r}_{1/4}^{(l,m)}$ can take on three possible meanings

$$\widetilde{r}_{1/4}^{(l,m=1)} = -2\widetilde{g}_{1/4}^{(l)}, \qquad \widetilde{r}_{1/4}^{(l,m=2,3)} = (-1 \pm i)\widetilde{g}_{1/4}^{(l)}$$

and coefficients $\widetilde{r}_{-1-5k/4}^{(n,l,m)}$, $k \in \mathbb{N}$ are uniquely determined.

The exponential additions of the third level can be presented as

$$\widetilde{\eta}(z) \equiv \widetilde{\eta}^{(l,m,p)}(z) = \widetilde{q}_{1/4}^{(l,m,p)} z^{1/4} + \frac{1}{4} z^{-1} + \sum_{k \in \mathbb{N}} \widetilde{q}_{-1-5k/4}^{(l,m,p)} z^{-1-5k/4},$$
$$n, l = 1, 2, 3, 4; \ m = 1, 2, 3; \ p = 1, 2 \tag{12.10}$$

where

$$\widetilde{q}_{1/4}^{(l,m,p=1,2)} = -\frac{3}{2} \widetilde{r}_{1/4}^{(l,m)} - 2\,\widetilde{g}_{1/4}^{(l)} \pm \frac{1}{2} i \sqrt{3\,(\widetilde{r}_{1/4}^{(l,m)})^2 + 8\,\widetilde{r}_{1/4}^{(l,m)}\,\widetilde{g}_{1/4}^{(l)} + 8\,(\widetilde{g}_{1/4}^{(l)})^2}$$

and coefficients $\widetilde{q}_{-1-5k/4}^{(l,m,p)}$, $k \in \mathbb{N}$ can be uniquely computed.

Taking into account three levels of exponential additions, we can write solutions $w(z)$ at $z \to \infty$ in the form

$$w(z) = \widetilde{\varphi}(z) +$$
$$+ \exp\left[\int dz \left(\widetilde{\zeta}^{(l)}(z) + \exp\left[\int dz \left(\widetilde{\xi}^{(l,m)}(z) + \exp\left(\int dz\, \widetilde{\eta}^{(l,m,p)}(z) \right) \right) \right] \right) \right]$$

where $l = 1, 2, 3, 4; \ m = 1, 2, 3; \ p = 1, 2.$

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13 Expansions to solutions of equation (1.1) near arbitrary point $z = z_0$.

In sections 3–9 we obtain all the expansions near points $z = 0$ and $z = \infty$. To obtain the expansions to solutions of equation (1.1) near arbitrary point $z = z_0$, we use the substitution $z' = z - z_0$. Then we get equation

$$w_{z'z'z'z'} - 5w^2 w_{z'z'} + 5w_{z'}w_{z'z'} - 5ww_{z'}^2 + w^5 - z'w - z_0 w - \alpha = 0 \quad (13.1)$$

Comparing this equation with equation (1.1) we see, that it has the additional term $-z_0 w$, therefore, the carrier of this equation contains all the points of the carrier of equation (1.1) and one additional point $Q_7 = (0,1)$. The convex hull of equation (13.1) coincides with quadrangle, presented at fig. 1, if $\alpha \neq 0$, and with triangle, given on fig. 3, if $\alpha = 0$. Point $Q_7 = (0,1)$ is the inner point, so it has no influence on asymptotic forms of solutions of equation (13.1).

If $z \to z_0$, then $z' \to 0$. Analyzing the expansions to solutions of equation (1.1), we see, that expansions near $z' = z - z_0 = 0$ correspond to apex Q_1 and edges Γ_1 and Γ_4 only.

As described above, there are four asymptotic forms, corresponding to apex Q_1 (see section 3) and one asymptotic form, corresponding to edge Γ_4 (see section 7). Power expansions, conforming to these asymptotic forms, can be generally written as

$$w(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + c_3(z - z_0)^3 +$$

$$+ \left(\alpha - 10\,c_1 c_2 + z_0\,c_0 - c_0^5 + 10\,c_2 c_0^2 + 5\,c_0 c_1^2 \right) \frac{(z - z_0)^4}{24} +$$

$$+ \left(z_0\,c_1 - 20\,c_2^2 + 40\,c_2 c_0 c_1 + 30\,c_3 c_0^2 + c_0 - 30\,c_1 c_3 + 5\,c_1^3 - 5\,c_0^4 c_1 \right) \frac{(z - z_0)^5}{120} + \ldots$$

where c_0, c_1, c_2 and c_3 are the arbitrary constants. If $c_1 = c_1 = c_2 = c_3 = 0$, this expansion corresponds to edge Γ_4, otherwise it conforms to apex Q_1.

Edge Γ_1 also generates four asymptotic forms, and, using the discussion, similar to presented in section 4, we obtain four families of expansions

$$w(z) = (z - z_0)^{-1} + c_1(z - z_0) + c_2(z - z_0)^2 +$$

$$+ \left(\frac{c_1^2}{4} - \frac{z_0}{20} \right)(z - z_0)^3 - \left(\frac{\alpha}{36} + \frac{1}{36} \right)(z - z_0)^4 + c_5(z - z_0)^5 +$$

$$+ \left(\frac{1}{160}\,z_0\,c_2 - \frac{7}{2880}\,c_1 - \frac{5}{576}\,c_1\alpha + \frac{1}{32}c_1^2 c_2 \right)(z - z_0)^6 + \ldots$$

$$w(z) = -2(z - z_0)^{-1} + c_1(z - z_0) + c_2(z - z_0)^2 +$$

$$+ \left(\frac{z_0}{10} - \frac{7{c_1}^2}{2} \right)(z - z_0)^3 + \left(\frac{1}{18} - \frac{5c_1 c_2}{2} - \frac{\alpha}{36} \right)(z - z_0)^4 +$$

$$+ c_5(z - z_0)^5 + \left(\frac{21}{4}{c_1}^2 c_2 - \frac{7}{80}z_0\, c_2 - \frac{71}{1440}c_1 + \frac{1}{36}c_1\alpha \right)(z - z_0)^6 + \dots$$

$$w(z) = -3(z - z_0)^{-1} - \frac{z_0}{60}(z - z_0)^3 + \left(\frac{\alpha}{84} - \frac{1}{28} \right)(z - z_0)^4 +$$

$$+ c_5(z - z_0)^5 + c_6(z - z_0)^6 - \frac{z_0^2}{4800}(z - z_0)^7 + \frac{z_0}{554400}\left(95\,\alpha - 299 \right)(z - z_0)^8 + \dots$$

$$w(z) = 4(z - z_0)^{-1} + \frac{z_0}{220}(z - z_0)^3 + \left(\frac{\alpha}{504} + \frac{1}{126} \right)(z - z_0)^4 +$$

$$+ c_5(z - z_0)^5 + \frac{17z_0^2}{5227200}(z - z_0)^7 + \frac{z_0}{2217600}\left(13 + 5\,\alpha \right)(z - z_0)^8 +$$

$$+ \left(\frac{137}{47500992}\alpha + \frac{1}{2992}z_0\, c_5 + \frac{37}{11875248} + \frac{25}{47500992}\alpha^2 \right)(z - z_0)^9 +$$

$$+ \frac{c_5}{15120}\left(6 + 5\,\alpha \right)(z - z_0)^{10} + c_{11}(z - z_0)^{11} + \dots$$

where c_1, c_2, c_5, c_6 and c_{11} are the arbitrary constants.

All the expansions, presented in this section, converge for sufficiently small $|z|$.

14   Conclusion.

In this paper we have shown that the self-similar solutions of both the Savada-Kotera equation and the Kaup-Kupershmidt equation can be expressed via the solutions of one of the fourth-order analogues of the second Painlevé equation. By means of power geometry methods we have got all power and non-power asymptotic forms of solutions of studied equation and all power expansions, corresponding to these expansions.

Let us briefly review the obtained results.

Near $z = 0$ we have found four-parametric family $G_1^{(0)}1$, three-parametric families $G_1^{(0)}2$, $G_1^{(1)}1$, $G_1^{(1)}2$, two-parametric families $G_1^{(0)}3$, $G_1^{(1)}3$, $G_1^{(1)}4$, one-parametric family $G_1^{(0)}4$ and family $G_4^{(1)}1$(if $\alpha \neq 0$). Families $G_1^{(0)}2$, $G_1^{(0)}3$, $G_1^{(0)}4$ and $G_4^{(1)}1$ are the special cases of $G_1^{(0)}1$.

Similar expansions have been obtained near arbitrary point $z = z_0 \neq \infty$.

Near point $z = \infty$ we have found families of expansions $G_2^{(1)}n$ ($n = 1, 2, 3, 4$) and $G_3^{(1)}1$ (if $\alpha \neq 0$). For each of these expansions three levels of exponential additions have been computed. If $\alpha = 0$, four families of non-power asymptotic forms near $z = \infty$ have been calculated too.

Expansions $G_2^{(1)}n$ ($n = 1, 2, 3, 4$) were found earlier [3,7], while all other expansions are the new ones.

The comparison between expansions, obtained in this paper, and ones, corresponding to the Painlevé equations [25–29] and their other fourth-order analogues [30, 31], confirms the hypothesis that the fourth-order equation (1.1) determines new transcendental functions.

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